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DESIGN OF OPTIMAL TRACKING DISCRETE CONTROL SYSTEM.(U)

JUL 79 Y BAR-NESS , J FEINSTEIN AFOSR-77-3063

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DESIGN OF OPTIMAL TRACKING DISCRETE CONTROL SYSTEM+

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DESIGN OF OPTIMAL TRACKING DISCRETE

CONTROL SYSTEM

Y. Bar-Ness and J. Feinstein

Abstract

Output feedback with a series compensator is considered for controlling SISO discrete systems. MSE criterion of design is used in tracking a given input sequence. The plant can be unstable and/or nonminimum phase, while the compensator is constrained to be physically realizable and closed-loop configuration is both stable and physically realizable. Using variational methods the required compensator is found, and shown to be the necessary and sufficient optimal solution.

I. Introduction.

The design of a discrete system to track a given input sequence is an important problem. The use of the minimum squared error (MSE) as a criterion of performance in such problems is well accepted. Kue[1] in treating this problem, did not take into consideration either the system configuration or the plant properties, so that in his solution the closed-loop may be unstable and the compensator obtained might not be physically realizable.

In this paper, output feedback together with a series compensator is considered for controlling the system to obtain the desired objective. This procedure is particularly suitable if the required compensator is constrained to be physically realizable(*) and the resultant closed-loop system is to be both stable and physically realizable.

Consider the physically realizable plant transfer function

$$G(z) = \frac{p^+(z)p^-(z)}{q^+(z)q^-(z)}$$

where p^+ and q^+ are polynomials whose zeros are located inside the unit circle and p^- and q^- are polynomials whose zeros are located outside this circle.

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*System is physically realizable iff the difference between the degrees of its transfer function's denominator and nominator, $\ell_{(\cdot)}$, is not less than zero.

For the given input sequence U_k and its corresponding output sequence W_k the squared error is given by,

$$J = \sum_{k=0}^{\infty} e_k^2 = \sum_{k=0}^{\infty} (U_k - W_k)^2. \tag{2}$$

Using Parseval's theorem, we get

$$J = \frac{1}{2\pi j} \oint E(z)E(z^{-1})\frac{dz}{z} \tag{3}$$

where

$$E(z) = U(z) - W(z) = \sum_{k=0}^{\infty} e_k z^{-k} \tag{4}$$

is the z transform of the error sequence.

Denoting the closed-loop transfer function by $T(z)$, we have

$$E(z) = (1-T(z))U(z) \tag{5}$$

with

$$T(z) = \frac{G(z)D(z)}{1+G(z)D(z)} \tag{6}$$

where $D(z)$ is the compensator transfer function that has to be obtained such that the squared error is minimized.

To solve the minimization problem, it is possible to use the Wiener filter approach and find $T(z)$. $T(z)$ may even be required to be a physically realizable solution. Nevertheless, with $T(z)$ available, the required compensator is obtained from equation (6)

$$D(z) = \frac{1}{G(z)} \frac{T(z)}{1-T(z)} . \tag{7}$$

Clearly, $D(z)$ must cancel some of the zeros or poles of the plant $G(z)$, which might be outside the unit circle, thus yielding an unstable closed-loop system. To cope with the problem of zero cancellation in the continuous case, Newton[2] and, later, Bongiorno[3] and Weston[4] restricted the plant to be asymptotically stable. Instead, as is done in this paper, it is reasonable to require that $D(z)$ is allowed to cancel only those poles and zeros of $G(z)$ which are inside the unit circle. That is

$$D(z) = \frac{q^{+}(z)\alpha(z)}{p^{+}(z)\beta(z)} \tag{8}$$

where $\alpha(z)$ and $\beta(z)$ are polynomials, yet to be determined.

To obtain these polynomials we define an appropriate variational problem on a space of polynomials such that it takes into account the requirements that $D(z)$ be physically realizable and that $T(z)$ is both stable and physically realizable. From the solution of the variation problem we obtain the necessary and sufficient condition which $D(z)$ - in particular $\alpha(z)$ and $\beta(z)$ - has to satisfy. It turns out that for $D(z)$ to be the required optimal solution, $\alpha(z)$ and $\beta(z)$ must satisfy some polynomial equation whose parameters are easily obtained from the input transform $U(z)$ and the plant $G(z)$ - see Theorem 4.6. Furthermore, we show that under a certain polynomial degree condition, the solution of this polynomial equation, $\alpha(z)$ and $\beta(z)$, is unique - see Lemma 4.4. With this, $D(z)$ is the required optimal compensator - see Theorem 4.7. Finally, we calculate the value of the optimal performance for comparison purpose. Two examples supplement the theoretical results.

Although we consider the case of a given input sequence, nevertheless, it is possible to extend this work to the case when the input includes noise or distrubances with a given spectral density.

Youla et al[5] treated a related problem for the continuous case, and (as in this work) the plant was allowed to be unstable and/or nominimum phase. In addition to the fundamental difference between the discrete and continuous systems, Youla's work differs from ours in several respects. The approaches are quite different. Youla's paper uses spectral factorization and some of the implicit assumption, might be rather restrictive (e.g. assumption 6). Secondly the resultant controller might be physically unrealizable depending on the given input or plant, a situation which might also be considered restrictive, particularly if flexible implimentation of this controller is required (see [5] foortnote on page 7). Finally, Theorem 1 of Youla constitutes only a necessary condition. Another reference which may be mentioned is Volgin[6], who discussed the problem without adequately solving it.

II. Stating the Problem.

Consider the discrete control system with output feedback and a series compensator. Let the plant $G(z)$ be given by equation (1). For a given input sequence U_k, it is required to find the polynomials $\alpha(z)$ and $\beta(z)$ of the compensator $D(z)$ (equation (8)), such that:

(1) $D(z)$ is physically realizable.

(2) The closed-loop transfer function $T(z)$ (equation (6)) is stable and physicall realizable.

(3) The sequared error between the given input sequence U_k and its corresponding output sequence W_k is minimized. (Equivalently, the cost functional of equation (3) is minimized.)

III. On the Constraints of the Variational Problem.

First, we consider some results related to the constraints of the variational problem. In terms of the polynomials $\alpha(z)$ and $\beta(z)$, the closed-loop transfer function is given by

$$T(z) = \frac{p^{-}(z)\alpha(z)}{v(z)} \tag{9}$$

where,

$$v(z) = p^{-}(z)\alpha(z) + q^{-}(z)\beta(z). \tag{10}$$

Propcsition 3.1: For $T(z)$ to be physically realizable it is necessary and sufficient that the degree δ_v of the polynomial $v(z)$ in equation (10) satisfies

$$\delta_v = \max(\delta_{p^{-}\alpha}, \delta_{q^{-}\beta}) \tag{11}$$

Proof: From equation (9), if $T(z)$ is physically realizable, then

$$\begin{aligned} \ell_T \triangleq \delta_v - \delta_{p^{-}\alpha} &\geq 0 \\ \Rightarrow \quad \delta_v &\geq \delta_{p^{-}\alpha}. \end{aligned} \tag{12}$$

From equation (10)

$$\begin{aligned} \delta_v &= \max(\delta_{p^{-}\alpha}, \delta_{q^{-}\alpha}) \quad \text{if} \quad \delta_{p^{-}\alpha} \neq \delta_{q^{-}\beta} \\ &\leq \delta_{p^{-}\alpha} = \delta_{q^{-}\beta} \quad \text{if} \quad \delta_{p^{-}\alpha} = \delta_{q^{-}\beta} \end{aligned}$$

and together with equation (12) the result follows. The other direction of our claim is obvious.

<u>Proposition 3.2</u>: Assume that the plant $G(z)$ is physically realizable, then

(a) For both the closed-loop transfer function $T(z)$ and the compensator $D(s)$ to be physically realizable it is necessary that

$$\delta_V = \delta_{q^- \beta} \geq \delta_{q^- \alpha}, \tag{13}$$

in which case,

$$\ell_T = \ell_G + \ell_D \tag{14}$$

where $\ell_{(\cdot)}$ is the excess denominator-nominator zeros.

(b) On the other hand, if

$$\delta_V = \delta_{q^- \beta} \geq \delta_{p^- \alpha} \tag{13}$$

and

$$\ell_T \geq \ell_G. \tag{15}$$

Then both $T(z)$ and $D(z)$ are physicall realizable.

<u>Proof</u>: (a) Using the fact that $G(s)$ is physically realizable we have from equation (1)

$$\ell_G \triangleq \delta_{q^+} + \delta_{q^-} - (\delta_{p^+} + \delta_{p^-}) \geq 0 \tag{16}$$

Similarly $D(z)$ is physically realizable implies

$$\ell_D \triangleq \delta_{p^+} + \delta_\beta - (\delta_{q^+} + \delta_\alpha) \geq 0. \tag{17}$$

These two equations together yield

$$\delta_{q^-} + \delta_\beta = \delta_{q^-\beta} \geq \delta_{p^-\alpha} = \delta_{p^-} + \delta_\alpha . \tag{18}$$

But, by Proposition 3.1, for $T(z)$ to be physically realizable it is necessary that

$$\delta_v = \max(\delta_{p^-\alpha} \; \delta_{q^-\beta}) \tag{19}$$

and, together with equation (18), the claim follows.

Furthermore, adding equations (16) and (17), we have

$$\ell_G + \ell_D = \delta_{q^-\beta} - \delta_{p^-\alpha} = \delta_v - \delta_{p^-\alpha} \triangleq \ell_T .$$

(b) From equation (13) and Proposition 3.1, $T(z)$ is physically realizable. Also, by equation (13)

$$\ell_T = \delta_{q^-} + \delta_\beta - (\delta_{p^-} + \delta_\alpha).$$

and together with equation (15), using equation (16) for ℓ_G, yield

$$\delta_{q^-} + \delta_\beta - (\delta_{p^-} + \delta_\alpha) \geq \delta_{q^+} + \delta_{q^-} - (\delta_{p^+} + \delta_{p^-}) \geq 0$$

$$\Rightarrow \quad \delta_\beta + \delta_{p^+} - (\delta_\alpha + \delta_{q^+}) \geq 0.$$

Hence, using equation (8), $D(z)$ is physically realizable.

IV. The Solution of the Variational Problem

For any given polynomial $x(z)$ of degree δ_x, let

$$\tilde{x}(z) \triangleq z^{\delta_x} x(z^{-1}). \tag{20}$$

If $x(z) = \sum_{i=0}^{n} a_i z^i$, $a_n \neq 0$ $\delta_x = n$, then clearly

$\tilde{x}(z) = z^n \sum_{i=0}^{n} a_i \bar{z}^i = \sum_{i=0}^{n} a_i z^{n-i}$. Also, if $x(z)$ has no zeros at $z = 0$ (i.e., $a_0 \neq 0$), then

$$\delta_{\tilde{x}} = \delta_x. \tag{21}$$

Proposition 4.1: Let $\Phi(z)$ be a rational function of z whose aero and poles are symmetrically located with respect to the unity circle, i.e.,

$$\Phi(z) = U(z)U(z^{-1}).$$

Then, it is possible to factor $\Phi(z)$

$$\Phi(z) = \frac{\tau^{+}(z)\tau^{-}(z)}{\rho^{+}(z)\rho^{-}(z)} \tag{22}$$

where the zeros of τ^{+} and ρ^{+} are located inside the unit circle and those of τ^{-} and ρ^{-} are outside it, such that

$$\tilde{\tau}^{+}(z) = \tau^{-}(z) \tag{23}$$

$$\tilde{\rho}^{+}(z) = \rho^{-}(z) \tag{24}$$

and

$$\delta_{\tau^{+}} = \delta_{\rho^{+}} \tag{25}$$

Proof: See Appendix A, claim (A-1).

We are now in a position to set the proper variational problem. Substituting equation (5) into equation (3) yields

$$J = \frac{1}{2\pi_j} \oint (1-T(z))(1-T(z^{-1}))\Phi(z)\frac{dz}{z} \tag{26}$$

where $T(z)$ is given by equations (9) and (10) and $\Phi(z)$

by equation (22). Therefore,

$$J = \frac{1}{2\pi j} \oint \frac{q^-(z)\beta(z)}{v(z)} \frac{q^-(z^{-1})\beta(z^{-1})}{v(z^{-1})} \frac{\tau^+(z)\tau^-(z)}{\rho^-(z)\rho^-(z)} \frac{dz}{z} .$$

Using equations (20) and (21) we have

$$J = \frac{1}{2\pi j} \oint \frac{q^-(z)\beta(z)}{v(z)} \frac{\tilde{q}^-(z)\tilde{\beta}(z)}{\tilde{v}(z)} \frac{\tau^+(z)\tau^-(z)}{\rho^+(z)\rho^-(z)} \frac{dz}{z} . \tag{27}$$

Hence, J can be considered as a functional on the linear space P of pairs of polynomials $(\alpha(z),\beta(z))$, which is defined in Appendix B. With the norm

$$||(\alpha,\beta)|| = \max\left(\sum_{i=0}^{\delta_\alpha}|\alpha_i|, \sum_{i=0}^{\delta_\beta}|\beta_i|\right) \tag{28}$$

defined on P, it can be shown that (see Appendix B) J is a differentiable functional on P whose first variation is given by

$$J[h_1,h_2] = \frac{-1}{\pi j} \oint \frac{q^-(z)\beta(z)}{v(z)} \frac{\tilde{p}^-(z)\tilde{q}^-(z)}{[v(z)]^2} \frac{\tau^+(z)\tau^-(z)}{\rho^+(z)\rho^-(z)} z^{\ell T}(\tilde{\beta}(z)\tilde{h}_1(z) - \tilde{\alpha}(z)\tilde{h}_2(z))\frac{dz}{z} \tag{29}$$

where h_1 and h_2 are the variation of α and β.

<u>Lemma 4.2</u>: Consider the linear space P, defined in Appendix B with the norm of equation (28). If

$$\delta_\alpha \geq \delta_{q^-} + \delta_{\tau^+} \tag{30}$$

and

$$\delta_v = \delta_{q^-\beta} \geq \delta_{p^-\alpha} \tag{31}$$

then for $(\alpha,\beta) \varepsilon P$ to be extremum of the cost functional of equation (26), it is necessary that

$$c(z) = \frac{q^-(z)\beta(z)\tilde{p}^-(z)\tilde{q}^-(z)\tau^+(z)\tau^-(z)z^{\ell_T}}{v(z)[\tilde{v}(z)]^2\rho^+(z)\rho^-(z)z} \tag{32}$$

where

$$v(z) = p^-(z)\alpha(z) + q^-(z)\beta(z) \tag{33}$$

has no poles inside the unit circle.

<u>Proof</u>: For J to have an extremum at $(\alpha,\beta) \varepsilon P$, the variation of equation (29) must vanish for every polynomial increment $h_1(z)$ and $h_2(z)$. Define

$$A(z) \triangleq \tilde{p}^-(z)\tilde{q}^-(z)\beta(z)\tau^+(z)\tau^-(z)p^-(z)z^{\ell_T}/d(z)$$

and

$$B(z) \underline{\Delta} zv(z)\varphi^{+}(z)/d(z)$$

so that $A(z)$ and $B(z)$ have no common zeros (i.e., $d(z)$ is the corresponding g.c.d.). Also, let

$$\mu(z) \underline{\Delta} (\tilde{\beta}(z)\tilde{h}_1(z) - \tilde{\alpha}(z)\tilde{h}_2(z)).$$

The polynomials $\tilde{v}(z)$ and $\rho^{-}(z)$ have no zeros inside the unit circle and can be disregarded. Clearly,

$$\delta_B \leq 1 + \delta_v + \delta_{\rho^+}. \tag{34}$$

For a given $\alpha(z)$ and $\beta(z)$, $h_1(z)$ and $h_2(z)$ can be selected such that $\mu(z)$ could have zero of order $\nu \leq \delta_\alpha + \delta_\beta$ anywhere in the complex plane, and particularly, inside the unit circle. From equations (30), (31), (34) and since $\delta_{\rho^+} = \delta_{\tau^+}$, by equation (25) we deduce that $\delta_\mu \geq \delta_B - 1$. Applying Lemma B-2 our claim follows.

<u>Lemma 4.3</u>: Assume that the closed-loop transfer function $T(z)$ is stable; then for every $\ell_T \geq 0$, for $c(z)$ to be analytic inside the unit circle, it is necessary that

$$v(z) = \tau^{+}(z)\tilde{p}^{-}(z)\tilde{q}^{-}(z)z^{\ell_T} \tag{35}$$

$$\beta(z) = z\rho^{+}(z)\pi(z) \tag{36}$$

where $\pi(z)$ is some polynomial which is together with $\alpha(z)$ must satisfy the following equation

$$p(z)\alpha(z) + q^{-}(z)\cdot z\rho^{+}(z)\pi(z) = \tau^{+}(z)p^{-}(z)q^{-}(z)z^{\ell_T}. \tag{37}$$

If, furthermore,

$$\delta_v = \delta_{q^-\beta} \geq \delta_{p^-\alpha} \tag{38}$$

then it is necessary that

$$\delta_\pi = \delta_{p^-} + \ell_T - 1 \tag{39}$$

and

$$\delta_\alpha = \delta_{\tau^+} + \delta_{q^-} \tag{40}$$

where $\ell_T \triangleq \delta_{q^-\beta} - \delta_{p^-\alpha}$.

On the other hand, if $\alpha(z)$ and $\pi(z)$ are a polynomial pair solution to equation (37), then with $\beta(z)$ as in (36), $c(z)$ is analytic inside the unit circle.

Proof: For $T(z)$ to be stable $v(z)$ must contain only zeros inside the unit circle. Since ℓ_T might be zero then for $c(z)$ to be analytic inside the unit circle we must have

$$v(z)z\rho^{+}(z)\pi(z) = \beta(z)\tilde{p}^{-}(z)\tilde{q}^{-}(z)\tau^{+}(z)z^{\ell_T} \tag{41}$$

for some polynomial $\pi(z)$. Substituting equation (33), and rearranging terms, we have

$$\begin{aligned}\beta(z)[\tilde{p}^{-}(z)\tilde{q}^{-}(z)\tau^{+}(z)z^{\delta_T} - zq^{-}(z)\rho^{+}(z)\pi(z)]\\ = p^{-}(z)\alpha(z)\pi(z)\rho^{+}(z)z.\end{aligned} \tag{42}$$

Since $\beta(z)$ and $p^{-}(z)\alpha(z)$ are relatively prime and $\pi(z)$ is arbitrary, equation (36) follows. And consequently, from equation (42) $\pi(z)$ must satisfy equation (37). Substituting equation (36) in equation (37) results in equation (35).

Furthermore, using equation (38) we have

$$\delta_\beta = \delta_{v^-} - \delta_{q^-}. \tag{43}$$

Since $p^{-}(z)$ and $q^{-}(z)$ have no zeros inside the unit circle, $\delta_{\tilde{p}^{-}} = \delta_{p^-}$ and $\delta_{\tilde{q}^{-}} = \delta_{q^-}$, and from equation (35) we have

$$\delta_v = \delta_{\tau^+} + \delta_{p^-} + \delta_{q^-} + \delta_T. \tag{44}$$

Also, from equation (36) we have

$$\delta_\beta = 1 + \delta_{\phi^+} + \delta_\pi. \tag{45}$$

Combining equations (43), (44) and (45) and using equation (25), equation (39) follows. Also, from equation (9)

$$\delta_T \triangleq \delta_v - \delta_{p^-\alpha},$$

and substituting for δ_v from equation (44), equation (40) follows after simple manipulations.

On the otherhand, if we pick $\pi(z)$ and $\alpha(z)$ to be a solution of equation (37) and take $\beta(z)$ as in equation (36), then clearly $v(z)$ will have the form of equation (35) and using equation (32) $c(z)$ is obviously analytic inside the unit circle.

Lemma 4.4: The solution pair of polynomials $\pi(z)$ and $\alpha(z)$ of equation (37),

$$p^-(z)\alpha(z) + q^-(z)z\rho^+(z)\pi(z) = \tau^+(z)\tilde{p}^-(z)\tilde{q}^-(z)^{\ell_T} \quad (37)$$

which satisfy

$$\delta_\pi = \delta_{p^-} + \delta_T - 1 \quad (39)$$

$$\delta_\alpha = \delta_{\tau^+} + \delta_{q^-}$$

is unique for all $\ell_T \geq \delta_G$

Proof: Assume that $\tau^+(z)\tilde{p}^-(z)\tilde{q}^-(z)z^{\ell_T}$ and $z\rho^+(z)$ are relatively prime. To solve equation (37) for the unknown polynomials $\alpha(z)$ and $\pi(z)$, notice that the number of unknowns (the coefficients of these polynomials) is $n_1 = \delta_\pi + \delta_\alpha + 2$. Using equations (39) and (40)

$$n_1 = \delta_{p^-} + \delta_T + \delta_{\tau^+} + \delta_{q^-} + 1. \tag{41}$$

On the otherhand, the number of scalar equations obtained by equating the coefficients of different powers is given by the degree of the polynomial on the left

$$n_2 = \delta_{\tau^+} + \delta_{\tilde{p}^-} + \delta_{\tilde{p}^-} + \delta_T + 1. \tag{42}$$

Since, $\delta_{\tilde{p}^-} = \delta_{p^-}$ and $\delta_{\tilde{q}^-} = \delta_{q^-}$ it follows that $n_1 = n_2$.

Now, the polynomials $p^-(z)$ and $q^-(z)z\rho^+(z)$ are relatively prime and hence their resultant is nonzero, and the matrix of the coefficients is nonsingular. his ensures the existence and uniqueness of the solution.

If the polynomials $\tau^+(z)\tilde{p}^-(z)\tilde{q}^-(z)z^{\ell_T}$ and $z\rho^+(z)$ are not relatively prime, we denote their g.c.d. by $d(z)$. Let

$$\alpha(z) = d(z)\alpha'(z)$$

$$z\rho^+(z) = d(z)\rho'(z). \tag{43}$$

$$\tau^{+}(z)\tilde{p}^{-}(z)\tilde{q}^{-}(z)z^{\ell_T} = d(z)v'(z). \tag{44}$$

Equation (37) thus becomes

$$p^{-}(z)\alpha'(z) + q'(z)\rho'(z)\pi(z) = v'(z).$$

This leaves the number of equations and unknowns equal and, therefore, the previous argument of existence and uniqueness for the pair $\alpha(z)$ and $\pi(z)$ follows.

Lemma 4.5: If

$$c(z) = \frac{q^{-}(z)\beta(z)p^{-}(z)q^{-}(z)\tau^{+}(z)\tau^{-}(z)z^{\ell_T}}{v(z)[v(t)]^{2}\rho^{+}(z)\rho^{-}(z)z} \tag{32}$$

is analytic inside the unit circle, then the second variation of the cost functional of equation (26) is given by

$$\mu^{2}J[h_1,h_2] = \frac{1}{\pi j}\oint \frac{\tilde{p}^{-}(z)\tilde{q}^{-}(z)}{[v(z)]^{2}}\,\Phi(z)z^{\ell_T}\,\frac{p^{-}(z)q^{-}(z)}{[\tilde{v}(z)]^{2}} \tag{45}$$

$$\cdot\ [\tilde{\beta}(z)\tilde{h}_1(z) - \tilde{\alpha}(z)\tilde{h}_2(z)][\beta(z)h_1(z) - \alpha(z)h_2(z)]\frac{dz}{z}$$

and is positive semidefinite.

Proof: Notice that the integrand of the first integral of equation (B-11) in Appendix B equals $c(z)/\tilde{v}(z)$. Therefore, it is analytic inside the unit circle and the integral is zero for every selection of $h_1(z)$ and $h_2(z)$. The second integral of equation (B-11) which is given in equation (45) can be rewritten in the form (see also equation (B-5))

$$\mu^2 J[h_1,h_2] = \frac{1}{nj} \oint \mu[T(z)]\mu[T(z^{-1})]\Phi(z)\frac{dz}{z} \tag{46}$$

Using Parseval's theorem, equation (46) can be represented as a sum of squares, hence $\mu^2 J[\ ]$ is positive semidefinite.

Finally, our results can be summarized as follows.

<u>Theorem 4.6</u> <u>(necessary condition)</u>: Let the input sequence for the discrete control system with output feedback be U_k and its z-transform $U(z)$. Define

$$\Phi(z) \triangleq U(z)U^{-1}(z) = \frac{\tau^+(z)\tau^-(z)}{\rho^+(z)\rho^-(z)} . \tag{22}$$

If the plant is physically realizable and given by

$$G(z) = \frac{p^+(z)p^-(z)}{q^+(z)q^-(z)} \tag{1}$$

then for the compensator

$$D(z) = \frac{q^+(z)\alpha(z)}{p^+(z)\beta(z)} \tag{2}$$

to be the optimal solution to the problem stated in Section II, it is necessary that

$$\beta(z) = z\rho^{+}(z)\pi(z) \tag{47}$$

$$v(z) = \tau^{+}(z)\tilde{p}^{-}(z)\tilde{q}^{-}(z)z^{\ell_T} \tag{48}$$

where

$$v(z) = p^{-}(z)\alpha(z) + q^{-}(z)\beta(z), \tag{49}$$

and $\pi(z)$ is a solution to the polynomial equation

$$p^{-}(z)\alpha(z) + q^{-}(z)z\rho^{+}(z)\pi(z) = \tau^{+}(z)\tilde{p}^{-}(z)\tilde{q}^{-}(z)z^{\ell_T}$$

with

$$\delta_\pi = \delta_{p^-} + \ell_T - 1, \quad (\ell \underline{\Delta} \delta_{q^-\beta} - \delta_{p^-\alpha})$$

$$\delta_\alpha = \delta_{\tau^+} + \delta_{q^-}$$

and where $\tilde{q}^{+}, \tilde{q}^{-}, \tilde{p}^{+}$ and $\tilde{p}^{-}$ are defined by equation (20).

<u>Proof</u>: Using Proposition 3.2, we have

$$\delta_v = \delta_{q^-\beta} > \delta_{p^-\alpha}$$

and using Lemma 4.2 and 4.3 our claim follows.

<u>Theorem 4.7</u> <u>(sufficient condition)</u>: For the discrete control system with output feedback and physically realizable plant as in Theorem 4.6, let $\pi(\alpha)$ and $\alpha(z)$ be the solution of

$$p^{-}(z)\alpha(z) + q^{-}(z)z\rho^{+}(z)\pi(z) = \tau^{+}(z)\tilde{p}^{-}(z)\tilde{q}^{-}(z)z^{\ell_T} \tag{37}$$

satisfying

$$\delta_\pi = \delta_{p^-} + \delta_T - 1$$

$$\delta_\alpha = \delta_{\tau^+} + \delta_{q^-}$$

where $\ell_T \geq 0$ and the different polynomials are specified in Theorem 3.9.

Let

$$\beta(z) = z\rho^{+}(z)\pi(z) \tag{50}$$

then using

$$D(z) = \frac{q^{+}(z)\alpha(z)}{p^{+}(z)\beta(z)} \tag{51}$$

as a series compensator, the closed-loop system is stable, the first variation will vanish and the second variation is positive semi-definite for all increments in $\alpha(z)$ and $\beta(z)$. If, furthermore, ℓ_T is chosen so that $\ell_T \geq \ell_G$, then $D(z)$ is also physically realizable.

Proof: First using Lemma 4.4, the pair $\alpha(z)$, and $\pi(z)$ is unique. From equation (8)

$$T(z) = \frac{p^-(z)\alpha(z)}{v(t)}$$

where $v(z)$ is

$$v(z) = \tau^+(z)\tilde{p}^-(z)\tilde{q}^-(z)z^{\ell_T} \qquad (52)$$

duc to the way $\beta(z)$ is selected. Hence, $T(z)$ is stable, and

$$\delta_v = \delta_{\tau^+} + \delta_{q^-} + \delta_{p^-} + \ell_T.$$

By Lemma 4.3, $c(z)$ (equation (32)) is analytic inside the in unit circle and, by equation (29), the first variation is zero. Also, by Lemma 4.5 the second variation of equation (53) is positive semidefinitc for all increments of the polynomials $\alpha(z)$ and $\beta(z)$.

Now, by our choice of $\pi(z),\alpha(z)$ and consequently of $\beta(z)$

$$\delta_{q^-} = \delta_{q^-} + 1 + \delta_{\rho^+} + (\delta_{p^-} + \ell_T - 1)$$

$$\delta_{p^-\alpha} = \delta_{p^-} + \delta_{\tau^+} + \delta_{q^-}.$$

Therefore, using $\delta_{\tau^+} = \delta_{\rho^+}$ and $\ell_T \geq 0$ we have

$$\delta_v = \delta_{q^-\beta} \geq \delta_{p^-\alpha} .$$

and if we choose $\ell_T \geq \ell_G$ then, by Proposition 3.2 the compensator $D(z)$ is physically realizable.

V. The Minimum Square Error.

For the unconstrained problem we could have used the Wiener solution to obtain the open-loop otpimal filter. For the case of simple poles, the optimal filter transfer function (our closed-loop transfer function) obtained using partial fraction expansion[1] has no excess of poles over zeros, i.e., $\ell_T = 0$.

Realizing this optimal Wiener filter $T(z)$ using series compensator configuration, we have that

$$D(z) = \frac{1}{G(z)} \frac{T(z)}{1-T(z)} .$$

Hence, if $\ell_G > 0$, $\ell_D = -\ell_G$ and $D(z)$ is physically unrealizable. By increasing ℓ_T, $D(z)$ becomes realizable although, obviously, the cost increases. One possible physically realizable solution can be obtained by choosing the minimal possible ℓ_T for which $\ell_D = 0$; that is $\ell_T = \ell_G$. Therefore, using Theorem 4.7,

$$D(z) = \frac{\alpha(z)q^{+}(z)}{zp^{+}(z)\rho^{+}(z)\pi(z)} \tag{54}$$

where $\alpha(z)$ and $\pi(z)$ are the solution pair of equation (37).

To find the minimum squared error, substitute equations (50) and (52) into equation (27) to get

$$J = \frac{1}{2\pi j} \oint \frac{q^{-}(z)\rho^{+}(z)\pi(z)}{\tau^{+}(z)\tilde{p}^{-}(z)\tilde{q}^{-}(z)} \cdot$$

$$\cdot \frac{q^{-}(z^{-1})\rho^{+}(z^{-1})\pi(z^{-1})}{\tau^{+}(z^{-1})\tilde{p}^{-}(z^{-1})\tilde{q}^{-}(z^{-1})} \frac{\tau^{+}(z)\tau^{-}(z)}{\rho^{+}(z)\rho^{-}(z)} \frac{dz}{z} \tag{55}$$

where by equation (20)

$$\tilde{q}^{-}(z^{-1}) = z^{-\delta_{q^{-}}} q^{-}(z)$$

$$q^{-}(z^{-1}) = z^{-\delta_{q^{-}}} \tilde{q}^{-}(z).$$

Also, it is possible to prove (see Appendix A, Claim (A-2)) that

$$\frac{\rho^{+}(z^{-1})\tau^{-}(z)}{\rho^{-}(z)\tau^{+}(z^{-1})} = 1.$$

and equation (54) thus becomes

$$J = \frac{1}{2\pi j} \oint \frac{n(z)n(z^{-1})}{\tilde{p}^-(z)\tilde{p}^-(z^{-1})} \frac{dz}{z} .$$

Again, with

$$\pi(z^{-1}) = 2^{-\delta_\pi} \tilde{\pi}(z)$$

and

$$p^-(z^{-1}) = 2^{-\delta_{p^-}}(z)$$

we have

$$J = \frac{1}{2\pi j} \oint \frac{\pi(z)\tilde{\pi}(z)}{p^-(z)\tilde{p}^-(z)z} \ell_G dz$$

Hence,

$$J = \sum \mathrm{Res}[\frac{\pi(z)\tilde{\pi}(z)}{z^{\ell_G} p^-(z)\tilde{p}^-(z)}] \tag{56}$$

where the summation is on all the residues inside the unit circle; i.e. at the poles of $z^{\ell_G} p^-(z)$.

VI. Examples.

(1) Consider the plant

$$G(z) = \frac{z-2}{(z-1,2)(z-0.5)}$$

and the transformed step input sequence

$$U(z) = \frac{z}{z-1} .$$

From equation (1) we get,

$$q^{+}(z) = z - 0.5, \quad q^{-}(z) = z - 1,2$$

$$p^{+}(z) = 1, \qquad p^{-}(z) = z - 2$$

and by equation (22)

$$\Phi(z) = \frac{z}{(z-1)(1-z)} .$$

Hence,

$$\tau^{+}(z) = z \qquad \tau^{-}(z) = 1$$

$$\rho^{+}(z) = z - 1 \quad \rho^{-}(z) = 1 - z.$$

Equation (37) becomes

$$(z-2)\alpha(z) + (z-1,2)z(z-1)\pi(z) = z\cdot(1-2z)(1-1,2z)\cdot z$$

where we chose $\ell_T = \ell_G = 1$. Dividing both sides by z, we have, in the notation of Lemma 4.4,

$$(z-2)\alpha'(z) + (z-1,2)(z-1)\pi(z) = z(1-2z)(1-1,2z)$$

The unique solution of this equation that satisfies $\delta_{\alpha'} = \delta_{\alpha} - 1 = \delta_{\tau^+} + \delta_{q^-} - 1 = 1$ and $\delta_{\pi} = \delta_{p^-} + \ell_T - 1 = 1$ is given by

$$\pi(z) = 5.7 + 2.4z$$

$$\alpha'(z) = 3.42 - 3.62z.$$

The compensator $D(z)$ (equation (54)) is, therefore,

$$D(z) = \frac{(z-1/2)(3.42-3.62z)}{(z-1)(5.7+2.4z)}$$

and using equation (56), the minimum square error is

$$J = \sum_{\substack{z=0\\z}} \text{Res}\left[\frac{(5.7+2.4z)(2.4+5.7z)}{z(z-2)(1-2z)}\right] = 17.31.$$

(2) Consider the plant

$$G(z) = \frac{(z-3/2)(z+2)}{(z-\alpha)(z-\beta)} \qquad |\alpha|,\ |\beta| < 1$$

and the transformed sequence

$$U(z) = \frac{(z-3)}{z-1}\ .$$

For this case we have,

$$q^+(z) = (z-\alpha)(z-\beta), \qquad q^-(z) = 1$$

$$p^+(z) = 1 \qquad , \qquad p^-(z) = (z-3/2)(z+2)$$

$$\tau^+(z) = 1 - 3z \qquad , \qquad \tau^-(z) = z - 3$$

$$\rho^+(z) = z - 1 \qquad , \qquad \rho^-(z) = 1 - z,$$

and the equation to solve

$$(z-3/2)(z+2)\alpha(z) + z(z-1)\pi(z) = (1-3z)(1 - \tfrac{3}{2}z)(1+2z)$$

with $\delta_\pi = 1$ and $\delta_\alpha = 1$. The unique solution can be found by equating different powers of z, yielding

$$\alpha(z) = -\frac{1}{3}(1+5z)$$

$$\pi(z) = \frac{2}{3}(11+16z).$$

With this, equation (54) yields the compensator

$$D(z) = -\frac{(z-\alpha)(z-\beta)(1+5z)}{2z(z-1)(11+16z)}$$

with minimum square error $J = 27.55$.

VII. Conclusions

Output feedback with a series compensator were considered, in controlling the SISO discrete system to track a given input sequence. MSE criterion of design was used in a variational problem whose solution yielded the required compensator transfer function $D(z)$. This compensator was constrained to be physically realizable and the resultant closed-loop system to be both stable and physically realizable.

It turns out that for $D(z)$ to be the required optimal solution, it must include two polynomials $\alpha(z)$ and $\beta(z)$ that have to satisfy some polynomial equation whose parameters are obtained from the input transform and the plant. On the other hand using the solution of this polynomial equation (which, under certain polynomial degree conditions is unique) $D(z)$ is the required optimal compensator. Two examples, that were worked out, present the detailed steps required.

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APPENDIX A

The z transform of the input sequence is given by

$$U(z) = z^{L} \frac{\prod_{i=1}^{I}(z-a_i)\prod_{k=1}^{K}(z-\alpha_k)}{\prod_{n=1}^{N}(z-b_n)\prod_{m=1}^{M}(z-\beta_m)} \tag{A-1}$$

where a_i, b_n, β_k and α_m are nonzero scalars, which satisfy

$$|a_i| \leq 1, \quad |b_n| \leq 1, \quad |\alpha_k| < 1 \quad \text{and} \quad |\beta_m| < 1.$$

L is some integer (positive or negative). For $U(z)$ to be causal we must have

$$N + M \geq L + I + K. \tag{A-2}$$

Hence,

$$\Phi(z) \triangleq U(z)U(z^{-1})$$

$$= z^{S} \frac{\prod_{i=1}^{I}(z-a_i)\prod_{k=1}^{K}(1-\alpha_k z)\prod_{K=1}^{K}(z-\alpha_k)\prod_{i=1}^{I}(1-a_i z)}{\prod_{n=1}^{N}(z-b_n)\prod_{m=1}^{M}(1-\beta_m z)\prod_{m=1}^{M}(z-\beta_m)\prod_{n=1}^{N}(1-b_n z)} \tag{A-3}$$

where

$$S = N + M - (I+K). \tag{A-4}$$

If we separate the poles and zeros to those inside, and those outside the unit circle, i.e., if

$$\Phi(z) = \frac{\tau^{+}(z)\tau^{-}(z)}{\phi^{+}(z)\phi^{-}(z)}$$

then we must have

$$\tau^{+}(z) = 2^{\frac{S}{2}[1+\operatorname{sgn} S]} \prod_{i=1}^{I}(z-a_i)\prod_{k=1}^{K}(1-\alpha_k z) \tag{A-5}$$

$$\tau^{-}(z) = \prod_{k=1}^{K}(z-\alpha_k)\prod_{i=1}^{I}(1-a_i z) \tag{A-6}$$

$$\rho^{+}(z) = z^{\frac{S}{2}[\operatorname{sgn} S-1]} \prod_{n=1}^{N}(z-b_n)\prod_{m=1}^{M}(1-\beta_m z) \tag{A-7}$$

$$\rho^{-}(z) = \prod_{m=1}^{M}(z-\beta_m)\prod_{n=1}^{N}(1-b_n z). \tag{A-8}$$

Notice that by equations (A-2) and (A-4), $S \geq L$, so that for $L \geq 0$, S is non-negative.

Claim A-1:

$$\tilde{\tau}^{+}(z) = \tau^{-}(z) \tag{A-9}$$

$$\tilde{\rho}^{+}(z) = \rho^{-}(z) \tag{A-10}$$

and

$$\delta_{\tau^{+}} = \delta_{\rho^{+}} . \tag{A-11}$$

<u>Proof</u>. By equation (20)

$$\tilde{\tau}^{+}(z) = z^{\delta_{\tau^{+}}}\tau^{+}(z^{-1})$$

and from equation (A-5)

$$\delta_{\tau^{+}} = \frac{S}{2}\,[1+\operatorname{sgn} S] + I + K. \tag{A-12}$$

Thus, using equation (A-5), we have

$$\tilde{\tau}^{+}(z) = z^{I+K}\prod_{i=1}^{I}(z^{-1}-a_i)\prod_{k=1}^{K}(1-\alpha_k z^{-1}),$$

and comparing with equation (A-6), equation (A-9) follows.

A similar proof applies to equation (A-10). *Finally*, substituting equation (A-4) into equation (A-12), we have

$$\delta_{\tau^{+}} = \frac{S}{2}\,[1+\operatorname{sgn} S] + N + M - S$$

$$= \frac{S}{2}\,[\operatorname{sgn} S-1] + N + M$$

and comparing with equation (A-7), equation (A-12) follows.

<u>Claim A-2</u>:

$$\frac{\rho^{+}(z^{-1})\tau^{-}(z)}{\rho^{-}(z)\tau^{+}(z^{-1})} = 1. \tag{A-13}$$

<u>Proof</u>. By equation (20)

$$\tau^+(z^{-1}) = z^{-\delta_{\tau^+}} \cdot \tilde{\tau}^+(z)$$

and using equation (A-9)

$$\tau^+(z^{-1}) = z^{-\delta_{\tau^+}} \cdot \tau^-(z). \qquad \text{(A-14)}$$

Similarly, applying equation (20) to $\rho^+(z^{-1})$ and using equation (A-10) we have

$$\rho^+(z^{-1}) = 2^{-\delta_{\rho^+}} \cdot \rho^-(z). \qquad \text{(A-15)}$$

Using equations (A-14) and (A-15), together with (A-11), yield our claim.

APPENDIX B

Consider the space P of pairs of polynomials $(\alpha(z),\beta(z))$ of degrees δ_α and δ_β respectively, i.e.

$$\alpha(z) = \sum_{i=0}^{\delta_\alpha} \alpha_i z^i \tag{B-1}$$

$$\beta(z) = \sum_{i=0}^{\delta_\beta} \beta_i z^i \tag{B-2}$$

(where α_i and β_i are numbers) such that

$$\delta_v = \delta_{q^-} + \delta_\beta \geq \delta_{p^-} + \delta_\alpha \tag{B-3}$$

where

$$v(z) = p^-(z)\alpha(z) + q^-(z)\beta(z).$$

Clearly, P is a linear space

The first variation of J.

Let $(h_1(z),h_2(z)) \in P$ be movements of $(\alpha(z),\beta(z))$. The corresponding increment of the functional J of equation (26) is,

$$\Delta J = J(\alpha+h_1,\beta+h_2) - J(\alpha,\beta)$$

$$= \frac{1}{2\pi j}\oint (1-T'(z))(1-T'(z^{-1}))\Phi(z)\frac{dz}{z}$$

$$- \frac{1}{2\pi j}\oint (1-T(z))(1-T(2^{-1}))\Phi(z)\frac{dz}{z}$$

where $T(z)$ is given by equation (10) and

$$T'(z) = \frac{p^-(z)(\alpha(z) + h_1(z))}{p^-(z)(\alpha(z) + h_1(z)) + q^-(z)(\beta(z) + h_2(z))} \quad \text{(B-4)}$$

Using Taylor's theorem we get

$$J = -\frac{1}{2\pi j}\oint (1-T(z))\mu[T(z^{-1})]\Phi(z)\frac{dz}{z}$$

$$- \frac{1}{2\pi j}\oint (1-T(z^{-1}))\mu[T(z)]\Phi(z)\frac{dz}{z}$$

$$- \frac{1}{2\pi j}\oint (1-T^*(z))\mu^2[T^*(z-^1)]\Phi(z)\frac{dz}{z}$$

$$- \frac{1}{2\pi j}\oint (1-T^*(z^{-1}))\mu^2[T^*(z)]\Phi(z)\frac{dz}{z}$$

$$+ \frac{1}{\pi j}\oint \mu[T^*(z)]\mu[T^*(z^{-1})]\Phi(z)\frac{dz}{z} + \ldots \quad \text{(B-5)}$$

In terms of the increment polynomials $h_1(z)$ and $h_2(z)$, it can be shown, that

$$\mu T(z) = \frac{p^-(z)q^-(z)}{[v(z)]^2}\,\mu[K(z)],$$

$$\mu^2 T(z) = \frac{-2p^-(z)q^-(z)}{[v(z)]^3}\,\mu[V(z)]\mu[K(z)] + \frac{p^-(z)q^-(z)}{[v(z)]^2}\,\mu^2[K(z)],$$

where

$$v(z) = p^-(z)\alpha(z) + q^-(z)\beta(z),$$

$$\mu v(z) = p^-(z)h_1(z) + q^-(z)h_2(z),$$

$$\mu K(z) = \beta(z)h_1(z) - \alpha(z)h_2(z),$$

and

$$\mu^2 K(z) = 0.$$

The (*) sybolize the fact that the corresponding terms are evaluated at $\alpha(z) + \theta_1 h_1(z)$ and $\beta(z) + \theta_2 h_2(z)$ with $0 \le \theta_1, \theta_2 \le 1$. It is easy to show by substituting z^{-1} for z that the first and the second integrals of equation (B-5) are identical and so are the third and fourth. Therefore;

$$J = \frac{-1}{\pi j} \oint \frac{q^-(z)\beta(z)}{v(z)} \Phi(z) \frac{p^-(z^{-1})q^-(z^{-1})}{[v(z^{-1})]^2} (\beta(z^{-1})h_1(z^{-1}) - \alpha(z^{-1})h_2(z^{-1}))\frac{dz}{z}$$

$$+ \frac{2}{\pi j} \oint \frac{q^-(z)\beta^*(z)}{v^*(z)} \Phi(z) \frac{p^-(z^{-1})q^-(z^{-1})}{[v^*(z^{-1})]^3} (p^-(z^{-1})h_1(z^{-1}) + q^-(z^{-1})h_2(z^{-1}))$$

$$\cdot (\beta^*(z^{-1})h_1(z^{-1}) - \alpha^*(z^{-1})h_2(z^{-1}))\frac{dz}{z}$$

$$+ \frac{1}{\pi j} \oint \Phi(z) \frac{p^-(z)q^-(z)}{[v^*(z)]^2} \cdot \frac{p^-(z^{-1})q^-(z^{-1})}{[v^*(z^{-1})]^2} (\beta^*(z^{-1})h_1(z^{-1}) - \alpha^*(z^{-1})h_2(z^{-1}))$$

$$\cdot (\beta^*(z)h_1(z) - \alpha^*(z)h_2(z))\frac{dz}{z} \qquad \text{(B-6)}$$

where

$$\alpha^*(z) = \alpha(z) + \theta_1 h_1(z)$$

$$\beta^*(z) = \beta(z) + \theta_2 h_2(z)$$

for some $0 \leq \theta_1, \theta_2 \leq 1$ and

$$v^*(z) = p^-(z)\alpha^*(z) + q^-(z)\beta^*(z) .$$

By the definition of the linear space $(\alpha^*(z), \beta^*(z))$ P. Also, $\delta_{v^*} = \delta_v$ and $\delta[p^-(z)h_1(z) + q^-(z)h_2(z)] = \delta_v$. Using equation (20), we have for the first integral in (B-6)

$$\Delta J_1 = \frac{-1}{\pi j} \oint \frac{q^-(z)\beta(z)}{v(z)} \Phi(z) \frac{\tilde{p}^-(z)\tilde{q}^-(z)}{[\tilde{v}(z)]^2} z^{\ell_T}(\tilde{\beta}(z)\tilde{h}_1(z) - \tilde{\alpha}(z)\tilde{h}_2(z))\frac{dz}{z} \qquad \text{(B-7)}$$

where we used the following

$$2\delta_{v^-} - (\delta_{p^-} + \delta_{q^-} + \delta_\alpha + \delta_\beta) = 2(\delta_{q^-} + \delta_\beta) - (\delta_{p^-} + \delta_{q^-} + \delta_\alpha + \delta_\beta)$$

$$= \delta_{q^-} + \delta_\beta - (\delta_{p^-} + \delta_\alpha).$$

$$= \ell_T.$$

For the second integral in (B-6) we have,

$$\Delta J_2 = \frac{2}{\pi j} \oint \frac{q^-(z)\beta^*(z)}{v^*(z)} \Phi(z) \frac{\tilde{p}^-(z)\tilde{q}^-(z)}{[\tilde{v}^*(z)]^3} (\tilde{p}^-(z)\tilde{h}_1(z) + \tilde{q}^-(z)\tilde{h}_2(z))$$

$$z^{\ell_T}(\tilde{\beta}^*(z)\tilde{h}_1(z) - \tilde{\alpha}^*(z)\tilde{h}_2(z))\frac{dz}{z} . \qquad \text{(B-8)}$$

Using the norm

$$||(h_1,h_2)|| = \max\left(\sum_{i=0}^{\delta_\alpha} |h_{1i}|, \sum_{i=0}^{\delta_\beta} |h_{2i}|\right),$$

it is possible to show that, $\frac{\Delta J_2}{||(h_1,h_2)||} \to 0$ as $||(h_1,h_2)|| \to 0$.

For the second integral in (B-6) we have,

$$\Delta J_3 = \frac{1}{\pi j} \oint \frac{p^-(z)q^-(z)}{[v^*(z)]^2} \Phi(z) \frac{\tilde{p}^-(z)\tilde{q}^-(z)}{[\tilde{v}^*(z)]^2} z^{\ell T}(\tilde{\beta}^*(z)\tilde{h}_1(z) - \tilde{\alpha}^*(z)\tilde{h}_2(z))$$

$$(\beta^*(z)h_1(z) - \alpha^*(z)h_2(z))\frac{dz}{z} . \qquad \text{(B-9)}$$

Similarly, $\Delta J_3/||(h_1,h_2)|| \to 0$ as $||(h_1,h_2)|| \to 0$. Therefore, the functional J is differentiable with

$$\mu J[h_1,h_2] = \frac{-1}{\pi j} \oint \frac{q^-(z)\beta(z)}{v(z)} \Phi(z) \frac{\tilde{p}^-(z)\tilde{q}^-(z)}{[\tilde{v}(z)]^2} z^{\ell T}(\tilde{\beta}(z)\tilde{h}_1(z) -$$

$$- \tilde{\alpha}(z)\tilde{h}_2(z))\frac{dz}{z} . \qquad \text{(B-10)}$$

The second variation of J. From equations (B-6), (B-7), (B-8) and (B-9) it is obvious that the second variation of J is

$$\mu^2 J[h_1,h_2] = \frac{2}{\pi j} \oint \frac{q^-(z)\beta(z)}{v(z)} \Phi(z) \frac{\tilde{p}^-(z)\tilde{q}^-(z)}{[\tilde{v}(z)]^3} z^{\ell T}(\tilde{p}^-(z)\tilde{h}_1(z) + \tilde{q}^-(z)\tilde{h}_2(z)$$

$$\times (\tilde{\beta}(z)\tilde{h}_1(z) - \tilde{\alpha}(z)\tilde{h}_2(z))\frac{dz}{z}$$

$$+ \frac{1}{\pi j} \oint \frac{p^-(z)q^-(z)}{[v(z)]^2} \Phi(z) \frac{\tilde{p}^-(z)\tilde{q}^-(z)}{[\tilde{v}(z)]^2} z^{\ell T}(\tilde{\beta}(z)\tilde{h}_1(z) - \tilde{\alpha}(z)\tilde{h}_2(z))$$

$$\times (\beta(z)h_1(z) - \alpha(z)h_2(z))\frac{dz}{z} . \qquad \text{(B-11)}$$

Lemma B-1. Let $A(z)$ and $B(z)$ be two given relatively prime polynomials. If

$$\oint \frac{A(z)}{B(z)} \mu(z)dz = 0$$

for any $\mu(z)$ of a given degree δ_μ, $\delta_\mu \geq \delta_B - 1$, then $A(z)/B(z)$ must be analytic inside the contour of integration.

<u>Proof</u>. By contradiction, assume that $B(z)$ has a pole of multiplicity ν, $\nu \leq \delta_B$ inside the contour of integration, i.e., $B(z) = C_1(z)(z-\alpha)^\nu$, $C_1(\alpha) \neq 0$ for some α. Choose $\mu(z)$, to have a pole at α of multiplicity $\nu - 1$, i.e., $\mu(z) = C_2(z)(z-\alpha)^{\nu-1}$, $C_2(\alpha) \neq 0$. Hence, by the residue theorem

$$\oint \frac{A(z)}{B(z)}\,\mu(z)dz = \frac{A(\alpha)C_2(\alpha)}{C_1(\alpha)} \neq 0$$

which contradicts our assumption.

References

[1] Kuo, B.C., Discrete-Data Control Systems, Englewood Cliffs, New Jersey, Prentice-Hall, 1970.

[2] Newton, G.C., L.A. Gould and J.F. Kaiser, Analytical Design of Linear Feedback Control, New York, Wiley, 1957.

[3] Bongiorno, J.J., "Minimum Sensitivity Design of Linear Multivariable Feedback Control Systems by Matrix Spectral Factorization", IEEE Trans. on Automatic Control, AC-14, 1969, pp. 665-673.

[4] Weston, J.E. and J.J. Bongiorno, "Extension of Analytical Design Techniques to Multivariable Feedback Control Systems", IEEE Trans. on Automatic Control, AC-17, 1972, pp. 613-620.

[5] Youla, P.C., H.A. Jabor and J.J. Bongiorno, "Modern Wiene-Hopf Design of Optimal Controller-Part I: The Single Input, Single Output Case", IEEE Trans. on Automatic Control, AC-21, 1972, pp. 3-13.

[6] Volgi, L.N., Elements of Theory of Computer Control, Sovyetskoye Radio, 1962.